Colors and Invariants Solutions

Pleasanton Math Circle: Middle School

Julian Xiao, Rohan Garg

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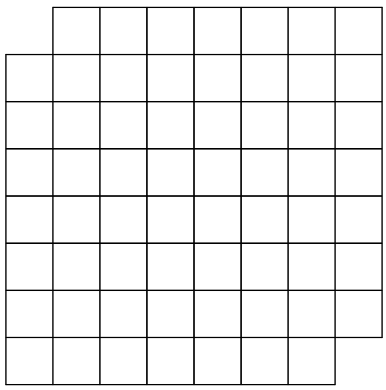
§1 Warm-Up

Problem 1.1. Suppose that I completely color one side of a sheet of paper in two colors, so that any point on this page will be one of the two colors. Prove that no matter how I color this piece of paper, I can always find two points an inch apart that are the same color. (Assume that the length and width of the page are much larger than 1 inch)

Proof. Consider an equilateral triangle with side length 1. There must be two vertices that are adjacent and the same color since we can't alternate if we have an odd number of vertices .

Problem 1.2. We take an 8×8 chessboard and remove two opposite corners like so:



We have 31 dominoes, and each domino is a 1×2 rectangle that can completely cover two neighboring squares of the chessboard. Is it possible to completely cover this chessboard with these 31 dominoes?

Proof. Color this in a normal chessboard fashion, with the two missing corners being black. Then we have 30 black tiles and 32 white tiles. Since one domino covers two adjacent cells, it must be one white and one black. But if we have 31 dominoes, we will cover 31 black and 31 white cells. This is impossible.

§2 Colors

The following problems involve colors. If the problem does not specifically mention the use of colors, then try to find a way to assign colors to different parts of the problem. For example, if the problem is about a chessboard, then try to assign colors to the squares. (In the more difficult problems, you may have to be more clever in how you assign colors!)

Problem 2.1. In every small cell of a 5×5 chess board sits a bug. At certain moment all the bugs crawl to neighboring (via a horizontal or a vertical edge) cells. Will there always be some cell that becomes empty?

Proof. Color this in a normal chessboard fashion (with all the corners being black). We have 13 black cells and 12 white cells. Since each bug moves to a cell adjacent to itself, it must switch colors. This means we will have 13 bugs that switched from black to white and 12 that switched from white to black. There will always be at least one missing black cell because we have a maximum of 12 bugs that can be on black cells.

Problem 2.2. Prove that it is impossible to cut a 10×10 chess board into 1×4 rectangles.

Proof. Think of this as a 5×5 board such that each cell in this 5×5 is a 2×2 square. Then color this in the exact same way as the previous problem, with more black than white. We can check that each 1×4 contains 2 of each color, so placing 25 of these 1×4s would get us 50 black and 50 white cells, while we have 52 blacks and 48 whites. This means it is impossible.

Problem 2.3. A rectangle is tiled with tiles of size 2×2 and 1×4. The tiles had been removed from the rectangle, and in the process one tile of size 2×2 was lost. We replaced it with a tile of size 1×4. Prove that it is impossible to tile the rectangle with the resulting collection of tiles.

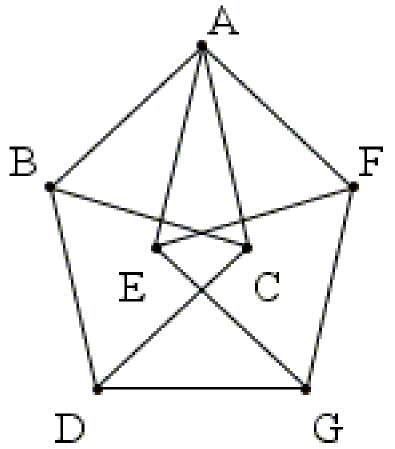
Proof. We use 4 colors now. Since we don't know the size of the grid, we can make up a rule that we follow to color. Starting from the top left tile, we color each diagonal the same color and when we move to the next diagonal, we use the next color (once we hit 4, we go back to 1). An example is:

1	2	3	4	1	2	3	4	1	2	3
2	3	4	1	2	3	4	1	2	3	4
3	4	1	2	3	4	1	2	3	4	1
4	1	2	3	4	1	2	3	4	1	2
1	2	3	4	1	2	3	4	1	2	3
2	3	4	1	2	3	4	1	2	3	4
3	4	1	2	3	4	1	2	3	4	1

We see that every 1×4 contains the numbers 1-4 while a 2×2 contains one number repeated twice, so if we remove it and add a 1×4 back, we will have an imbalance.

Problem 2.4. Consider Problem 1.2. This time, prove that I can always find two points an inch apart that are the same color if you have three colors to choose from.

Proof. Consider the following:



This diagram consists of 4 equilateral triangles and one equilateral pentagon. If any of A,B,C are the same color, then we are done. Similarly, if any of the B,C,D are the same color, we are done. So we assume they are all distinct. This means A and D are the same color, since there is only one color left to choose from after coloring B,C. Through a similar argument, we see that G has the same color as A. This means D and G are the same color, so we are done. □

§3 Invariants

Definition 3.1 (Invariant)**.** An invariant is a property that doesn't change after performing an operation. Try to spot an invariant in the following examples:

- We are given a polygon on the plane. An operation is defined as either rotating, translating, or reflecting it. What stays the same? (There are many answers)

  Proof. The area, angles, shape, etc stay congruent.

  A board has a bunch of numbers on it. An operation is defined as taking two numbers on the board and replacing them with the square root of their product. For example, 6 and 7 would be replaced with $\sqrt{42}$ and $\sqrt{42}$. What stays the same?

  Proof. The product of the numbers stays the same.

Problem 3.2. You will fill in each square in the following expression with either a + or − sign:

$$1\square 2\square 3\square 4\square 5\square 6\square 7\square 8\square 9\square 10$$

Prove that no matter how you choose to fill in the squares, this expression cannot equal 0.

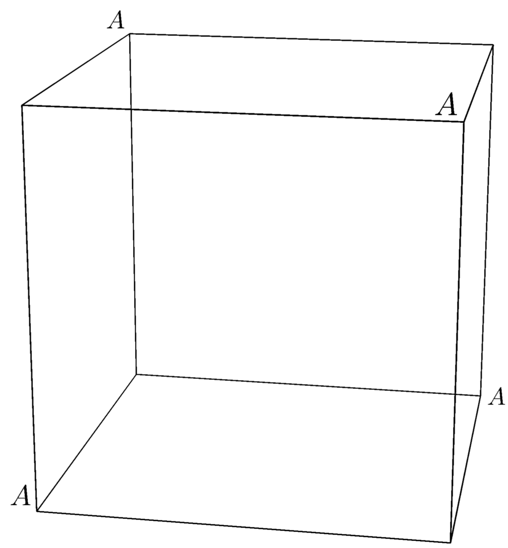
Proof. The invariant is the parity, no matter how we choose the symbols we must have an odd output. This is because if we group the terms we add and the terms we subtract, to get 0, these must be the same. We have $x - x = 0$, but then $x + x = 1 + 2 + \cdots + 10$, so $x = 55/2$. This is not an integer so it is not possible.

Problem 3.3. We write the numbers from 1 through 10 on a board. An operation consists of taking two numbers, say a and b, erasing them, and replacing them with $ab + a + b$. If we repeat this until there is one number left, what is this number?

Proof. The invariant is adding 1 to all the numbers and multiplying these new numbers stays the same. If the product of all the numbers+1 except the two numbers we changed (a,b) was P, then the board had product $P(a + 1)(b + 1)$. After applying the operation, we get $ab + a + b$, so the new product is $P(ab + a + b + 1)$. This is equal to $P(a + 1)(b + 1)$. Then our final number must satisfy this invariant as well. If it is x, then $x + 1 = (1 + 1)(2 + 1) \dots (9 + 1)(10 + 1)$, so $x = 11! - 1$.

Problem 3.4. Seven vertices of a cube are labeled 0, and the remaining vertex labeled 1. You're allowed to change the labels by picking an edge of the cube, and adding 1 to the labels of both of its endpoints. After repeating this multiple times, can you make all labels divisible by 3?

Proof. The answer is no. Label the cube in the following way:



The vertices labelled with A and the vertices not labelled with A are their own groups. We can see that any operation affects one vertex from each group. This means the difference of the sum of the labels in each group stays the same. At the start, the difference was 1 or -1 (depending on where the 1 is). But if all the labels were divisible by 3 some point later on, the difference would be divisible by 3. This means it is impossible.

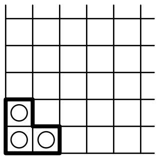
Problem 3.5. Challenge: Show that when a 6×6 square floor is tiled using 1×2 rectangular tiles, there is always a straight line which crosses the floor without cutting through any of the tiles.

Proof. First, we notice that such a line must be one of the 10 horizontal or vertical lines (gridlines). We claim that each of these lines crosses an even number of tiles. We know that the area on each side of the line is even, since one dimension is 6. But if it crossed an odd number of tiles, the remaining 1×2 tiles would have odd area, which is impossible. This means that each line crosses an even number of tiles. For contradiction, assume that each of the lines crosses at least one tile. Then each crosses at least 2, so we must have at least $10 \cdot 2 = 20$ tiles. We only have 18 which is a contradiction.

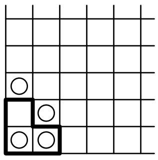
Problem 3.6. Challenge: A 10×10 square field is divided into 100 equal square patches, 9 of which are overgrown with weeds. It is known that during a year the weeds spread to those patches that have no less than two neighboring (i.e., having a common side) patches that are already overgrown with weeds. If a patch does not have at least two neighboring patches with weeds, then weeds will not spread to that patch. Prove that the field will never overgrow completely with weeds.

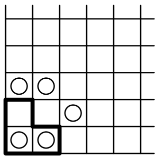
Proof. The key idea is that after each year, the perimeter of the infected region stays the same or decreases. This can be verified by checking some cases, which we skip here. The maximum perimeter of the 9 original squares is 36 while the perimeter of the whole field is 40, so it is not possible that at any time, the whole field was covered with weeds.

Problem 3.7. "Three Prisoners." Challenge:



Consider a grid that extends infinitely in the rightward and upward directions. At the start, three dots or "prisoners" are in the three cells in the bottom-left corner. These prisoners want to escape the prison (i.e. the outlined region), and they can move in the following way: if both the cell above and the cell to the right are empty, the prisoner can divide into two and move to those cells. Basically, replace the existing prisoner and fill in the cells above it and to the right of it. The two diagrams below show a possible sequence of two moves you can start with:





Is it possible for all prisoners to escape the prison? That is, is it possible for the three cells at the bottom-left to be empty? Once you have an answer, can you prove it?

Solution. The key is to assign number values to each of the cells in the grid as shown (remember that the grid is infinite, so this pattern continues):

1/32	1/64	1/128	1/256	1/512	1/1024
1/16	1/32	1/64	1/128	1/256	1/512
1/8	1/16	1/32	1/64	1/128	1/256
1/4	1/8	1/16	1/32	1/64	1/128
1/2	1/4	1/8	1/16	1/32	1/64
1	1/2	1/4	1/8	1/16	1/32

Notice that anytime a dot splits, the two new dots will take up two spaces each with a number half of the original space. So for example, a dot that starts on the 1/2 square will then split up and fill up two 1/4 squares. By numbering the whole grid in this manner, we ensure that no matter what moves we make, the sum of all the squares that the dots occupy will remain the same. We start off with a sum of 2 in the bottom three squares. Notice that for the prisoners to escape from the prison, it can only be possible if the outside spaces have a total of at least 2.

So now let's calculate the total value of the grid. The bottom row of the grid has values $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, which equals 2. Here is a brief explanation of why.

$$\begin{aligned}
S &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\
2S &= 2(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots) = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots \\
2S - S &= (2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots) - (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots) \\
S &= 2
\end{aligned}$$

$\square$

Using the same reasoning, we can find that the total value of the second-bottom row is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \frac{1}{2}$. The third-bottom row has a total of $\frac{1}{4}$ and the fourth-bottom row has a total of $\frac{1}{8}$ and so on. By taking the total of each row, we get something like this:

1/32	1/64	1/128	1/256	1/512	1/1024	=1/16
1/16	1/32	1/64	1/128	1/256	1/512	=1/8
1/8	1/16	1/32	1/64	1/128	1/256	=1/4
1/4	1/8	1/16	1/32	1/64	1/128	=1/2
1/2	1/4	1/8	1/16	1/32	1/64	=1
1	1/2	1/4	1/8	1/16	1/32	=2

Now let's add each of these sums together to get the total of all the numbers in this grid. We get $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$, which equals 4. Because all spaces on this grid have a total value of 4, the prison has a total value of 2, and everything outside of the prison has a total value of 2 as well. This means that if we want to vacate the prison by moving the prisoners outside, then we will need to have a dot in every single cell outside the prison! Because this is an infinite grid, it is simply impossible to fill every square outside the prison, so it is impossible for all prisoners to escape.